THE LOTKA-VOLTERRA MODEL IN N DIMENSIONS (for any N)

$$\frac{dx_i}{dt} = \alpha_i x_i \left(K_i - \sum_{j=1}^{N} \beta_{ij} x_j \right)$$

x_i : Species i (i=1,2,...,N) population

α_i : Growth rate

K_i : Carrying Capacity function for species i

β_{ij} : Competition coefficients between species i and δ

LOTKA-VOLTERRA MODEL APPLIED TO URBAN AND REGIONAL SCIENCES

Prey ↔ average income of studied micro area ↔ supply
Predator ↔ population of studied micro area ↔ demand

In sum: the population “catches” the average income

or

there are dynamical interactions between demand (predator) and supply (prey)

ANALYTICAL MODELS OF SPACE DEBRIS IN LOW EARTH ORBIT

Big objects: satellites ↔ prey
Small objects: debris ↔ predator

SIMPLE MODEL FOR THE EVOLUTION OVER TIME OF ALL ORBITING BODIES

$$\boxed{\frac{dN}{dt} = A + BN + CN^2}$$

N = number of orbiting bodies

A = "deposition" coefficient (it is articulately "modelled", with a series of experimental "sub-coefficients". It is connected to the number and type of rocket launches)

B = "removal" coefficient (it is "modelled" as follows):

$B = B_{atm} + S$, with

$$\begin{cases} B_{atm} = \text{fraction of bodies that fall under natural causes (residual atmosphere)} \approx -5.6 \times 10^{-3} \\ S = \text{fraction of bodies that fall under active removal systems} \approx 0 \end{cases}$$

N.B.
one could also immediately write - B*N* !

C = "collision" coefficient (it is "modelled" in a complex way, both by means of technical considerations, Kinetic theory of gases, and of experimental data)

MODEL FOR THE EVOLUTION OVER TIME OF TWO POPULATIONS OF SPACE BODIES

(SATELLITES – S – AND DEBRIS – F)

big objects small objects

$$\boxed{\begin{cases} \dfrac{dN_S}{dt} = A_S - B_S N_S - C_{S1} N_S^2 - C_{S2} N_S N_F \\ \dfrac{dN_F}{dt} = A_F - B_F N_F + C_{F1} N_S^2 + C_{F2} N_S N_F + C_{F3} N_F^2 \end{cases}}$$

In this case, one can view the problem as an interaction between two populations of animals that are, however, cannibals i.e., they are self-destructive populations both subject to "hunt" and to repopulation by an external agent.
One population can be seen as "prey" satellites, and the other as "predator" debris.
The preys-satellites are those bigger than a certain threshold, and they are hunted by the predator-debris, which are smaller. However, the species are cannibals i.e., there can be interaction between elements of the same species: debris have as preys not only the bigger satellites, but also the debris themselves. At the same time, bigger satellites can destroy "one of their own", but not a debris. Moreover, both species are subject to "hunting" by the Earth's atmosphere that acts upon then, and a deposition term is present in both of the species equations. This term will depend on the human activity of orbit launch.

EPIDEMIOLOGICAL MODEL (Kermack, McKendrick: 1927)

$$\begin{cases} \dfrac{dS}{dt} = -kSI & , S(0) = S_0 > 0 \\[2ex] \dfrac{dI}{dt} = kSI - hI & , I(0) = I_0 > 0 \\[2ex] \dfrac{dR}{dt} = hI & , R(0) = R_0 > 0 \end{cases}$$

HYPOTHESIS

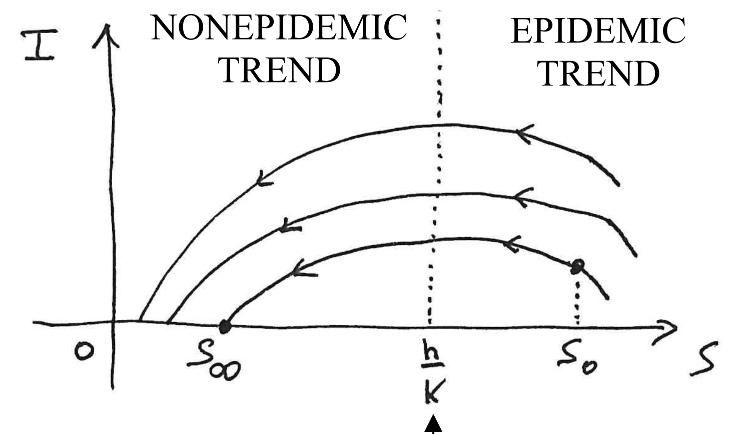
1) The infected immediately becomes contagious;
2) The disease gives immunity
3) N= S+I+R= constant i.e., births, deaths (for other causes) and migratory flows are disregarded

The first two equations do not depend on R: therefore, they can be solved separately e.g., by dividing the second by the first (assuming that S and I $\neq$ 0):

$$\frac{dI}{dS} = \frac{I(kS-h)}{-kSI} \Rightarrow \frac{dI}{dS} = -1 + \frac{h}{kS} \Rightarrow dI = \left[-1 + \frac{h}{kS}\right] dS \Rightarrow$$

$$\Rightarrow I(S) = -S + \frac{h}{k}\log S + c \qquad \begin{cases} \text{con le C.I.: } I_0 = -S_0 + \frac{h}{k}\log S_0 + c \Rightarrow \\ \Rightarrow c = I_0 + S_0 - \frac{h}{k}\log S_0 \end{cases}$$

Ultimately, the following trend is obtained:



$\boxed{S_0 < \frac{h}{k}}$ ← THRESHOLD VALUE → $\boxed{S_0 > \frac{h}{k}}$

NONEPIDEMIC SPREAD — (I(S) has the max) — EPIDEMIC SPREAD

Please note that:

$$\frac{dI}{dt} > 0 \text{ se } I(kS-h) > 0,$$

cioè, se

$$kS - h > 0 \Rightarrow S > \frac{h}{k}$$

In other terms, there is an epidemic trend or epidemic spread if I(t) is increasing.
On the contrary, if I(t) is decreasing, this means that the epidemic has been overcome

→ (not to have) (i.e., an epidemic trend of the contagious disease)

In order to have an infection, it must happen that

$$S_0 > \frac{h}{K} \Rightarrow \boxed{S_0 \frac{K}{h}} > 1 \Rightarrow R_0 > 1$$

$$\rightarrow \left(S_0 < \frac{h}{K} \Rightarrow R_0 < 1 \right)$$

Defining this quantity:

$R_0 = S_0 \frac{K}{h}$ { BASIC REPRODUCTION NUMBER

↓

net reproduction number of the disease

↓

reproductive rate of infection in the population

↓

it is an indicator of the maximum disease spread potential

↓

N.B. It is explained in paper nr.3 { Basically, this number represents, on average, how many individuals a contagious individual can infect

We observe that R_0 can be estimated (see the table) and these estimates are referred to a population **without** vaccine coverage. However, if we give the vaccine to part p of the population (i.e., a "proportion" p (→ fraction of vaccinated people) smaller than 1, of the population: for instance, if p=0.3 this means that 30% of the population gets the vaccine), this modifies S_0 that becomes $(1-p)S_0$. Therefore, in this case i.e., by vaccinating a part p of the population (to be defined) **there won't be an epidemic if**

$$(1-p)S_0 < \frac{h}{K} \Rightarrow (1-p)S_0\frac{K}{h} < 1 \Rightarrow$$

$$\Rightarrow \boxed{(1-p)R_0} < 1 \Rightarrow R_V < 1$$

{ is the condition in order to remove the infection i.e., so as not to spread the disease

Defining this quantity R_v: REAL REPRODUCTION NUMBER

↓

Since based on experimental data (pre-vaccination era) we know the values of R_0, we can estimate the vaccine coverage required to curb the infection:

For example	Article "Prisma"	Book Barnes, Fulford
Infection	R_0	R_0
Smallpox	5 – 7	4
Measles and pertussis (book)	12 – 8	16 – 18
Flu (a strain of)	2	3 – 4
Mumps	4	
Chickenpox		10 – 12

LORENZ ATTRACTOR (1917-2008)

Text translated from: G.BORGIOLI "modelli matematici di evoluzione ed equazioni differenziali" pp. 86-88, ED. CELID, 1996

We will look at one of the best-known mathematical models in nonlinear dynamics that show instances of chaotic behavior. Thanks to E.N.Lorenz's theorization, this model is a simple system of three autonomous first-order differential equations that exhibit quadratic non-linearity (as in the predator-prey model, for instance). The physical phenomenon described is the thermal convection of an incompressible, viscous fluid in a rectangular plane region (cell) positioned vertically (Fig.4.19).

The system of ordinary equations is obtained by starting from the "exact" model, which is built from a system of partial differential equations (the Navier-Stokes equations), where the variables (unknowns) include the velocity field of the fluid, temperature, etc. By applying a Fourier series expansion to the unknown functions and truncating the series, the system is derived.

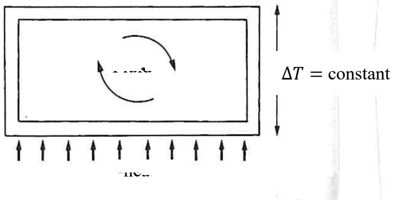


Fig.4.19

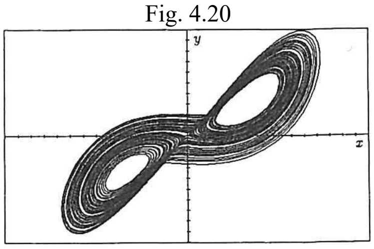
The partial differential equations are transformed into ordinary equations, where the unknowns are the coefficients of the Fourier expansion up to the considered order. In this case, we disregard the (infinite) terms beyond a certain order because we expect the solution (eventually, after a transient period) to oscillate only according to certain vibrational modes, while the remaining ones are essentially negligible.

$$\begin{cases} \dot{X} = -\sigma(X-Y) \\ \dot{Y} = RX - Y - XZ \\ \dot{Z} = XY - bZ \end{cases} \quad ; \qquad (4.29)$$

X is the coefficient of the first term for flow development (a scalar function that, under the assumption of a plane system invariant with respect to the third spatial coordinate, replaces the velocity vector field); Y is the first coefficient for temperature development, and Z describes temperature's vertical pattern. The coefficients σ (the Prandtl number) and R (the dimensionless Reynolds number) characterize the physical properties of the fluid, b is a parameter related to the geometric characteristics of the cell.
According to Lorenz's expectations (and those who preceded him in attempting to derive approximate models of the Navier-Stokes equations), equation (4.29) was supposed to be a "simple" model for studying atmospheric convection motions and, therefore, an important milestone in the development of "deterministic" weather forecasts.
For an extensive analysis of the properties of the system (4.29) (see 22, Chapter 11), aside from the equilibrium solution (0, 0, 0), one can easily determine other two additional equilibrium solutions. Keeping σ and b fixed, one can study their respective stability properties at the variation of R (increasing values of R correspond, for instance, to a greater thermal gradient between the two horizontal faces of the cell, leading to a more significant convective flow). For values $R \cong 28$, all equilibrium solutions become unstable, and for all the considered evolution solutions, an extreme sensitivity with reference to the initial data was observed (they exponentially move away from each other). However, for all the initial conditions considered, the evolution asymptotically converges to an invariant region of the phase space (X, Y, Z). This region exhibits an extremely complex geometric structure (known as **fractal**) and has been identified as the **chaotic Lorenz attractor** (often referred to as "butterfly wings" due to its characteristic shape). The consequences of this result for predicting a later stage of the system with fixed precision are rather catastrophic since, as mentioned earlier, the uncertainty (measurement error) in the initial state spreads to later stages with exponential growth over time.

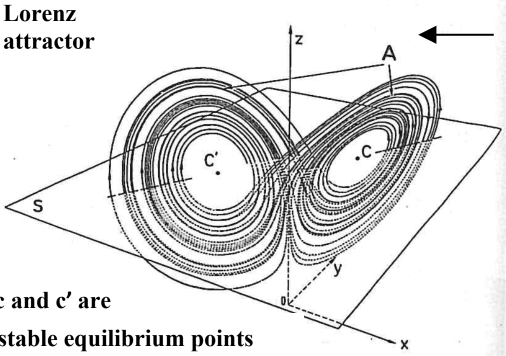
The resulting implications on weather forecasts, assuming that real air turbulence exhibits the same sensitivity to initial values, were discussed by Lorenz in his works from 1963 and 1964. In fig. 4.20 you can see the projection of the attractor onto the plane (X, Y) generated using PHASER [14].

Fig. 4.20



Some of the most defining aspects of nonlinear differential equations include the following: First, in general, it is not possible to explicitly obtain the solution of an initial value problem (IVP) for a nonlinear system in terms of elementary functions. Once the existence and uniqueness of the solution are established (see Chapter 5), a qualitative approach becomes the only viable method for understanding its properties, using both rigorous mathematical analysis and numerical simulations. mong the most interesting characteristics, already observable in two-dimensional systems, is the possibility of periodic asymptotic solutions (limit cycles), even in unforced systems (Van der Pol equation). Solutions of chaotic type can appear starting from dimension 3 (equation of forced pendulum, Duffing equation, Lorenz model).
This aspect of nonlinear dynamics has gained increasing popularity in recent times and has been presented here in a purely phenomenological manner. In this context, we observe that even in completely deterministic systems (to each possible initial data corresponds one and only one solution), determinism itself faces a crisis when chaotic solutions arise. Specifically, measurement errors in the initial conditions spread rapidly, making it impossible to achieve exact long-term predictions about the system's state.

Lorenz attractor



Figures taken from A.Vulpiani "Determinismo e Caos", Ed.Carocci, 2004

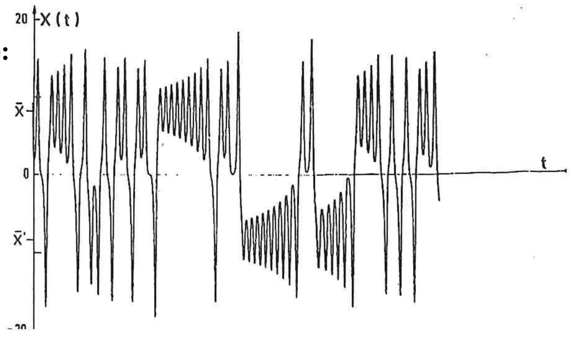
c and c′ are unstable equilibrium points

X(t) as a function of t for the Lorenz model with σ=10, b=8/3 and r=28

For example:
R=28
σ=10
b=$\frac{8}{3}$

or

R=15
σ=5
b=1



Partial Differential Equations: PDEs

PDEs $\Rightarrow$ infinite solutions

PDEs + complementary conditions $\Rightarrow$ one solution only (under regularity conditions for the unknown functions)

complementary conditions

initial conditions + boundary conditions for hyperbolic and parabolic equations (i.e., diffusion equation, wave equation)

t $$\frac{\partial}{\partial t}T(x,t) = \frac{k}{\rho c}\frac{\partial^2}{\partial x^2}T(x,t) \qquad \frac{\partial^2}{\partial t^2}A(x,t) = c^2\frac{\partial^2}{\partial x^2}A(x,t)$$

boundary conditions for elliptic equations (i.e., Poisson equation and Laplace equation)

$$\frac{\partial^2}{\partial x^2}\Phi(x,y) + \frac{\partial^2}{\partial y^2}\Phi(x,y) = -\frac{1}{\varepsilon_0}\rho(x,y) \qquad \frac{\partial^2}{\partial x^2}\Phi(x,y) + \frac{\partial^2}{\partial y^2}\Phi(x,y) = 0$$

Partial Differential Equations: PDEs

initial conditions + boundary conditions
for hyperbolic and parabolic equations
(i.e., diffusion equation, wave equation)

$$\frac{\partial}{\partial t}T(x,t)=\frac{k}{\rho c}\frac{\partial^2}{\partial x^2}T(x,t)$$

$$\frac{\partial^2}{\partial t^2}A(x,t)=c^2\frac{\partial^2}{\partial x^2}A(x,t)$$

It arises from a thermal balance when *T* is temperature. It is commonly known as heat equation or Fourier equation.
If T represents concentration, it describes the diffusion of a diluted substance (gas or liquid) in an homogeneous dispersing phasem, which is typical in solutions. In this case, it is referred to as Fick's equation, where mass diffusivity replaces thermal diffusivity (*k/(ρc)*).

It represents vibratory phenomena, waves that propagate. It was discovered by D'Alambert vibrating strings.
This equation describes transverse waves of small amplitude in a string, waves in an elastic membrane, sound waves and also electromagnetic waves in a vacuum.

Partial Differential Equations: PDEs

boundary conditions

for elliptic equations

(i.e., Poisson equation and Laplace equation)

$$\frac{\partial^2}{\partial x^2}\Phi(x,y)+\frac{\partial^2}{\partial y^2}\Phi(x,y)=-\frac{1}{\varepsilon_0}\rho(x,y)$$

It represents the non-homogeneous version of the Laplace equation. It can describe a membrane subject to a force density, a thermal field with source, an electric field in the presence of a charge density (it is important in many electrostatic problems, just as its homogeneous version is)

$$\frac{\partial^2}{\partial x^2}\Phi(x,y)+\frac{\partial^2}{\partial y^2}\Phi(x,y)=0$$

It is also known as potential equation. It can yield static solutions (stationary: solution that no longer depends on time) of the D'Alambert equation or the heat equation. It can describe stationary thermal fields, electrostatics problems in 2-D, 2-D motions of perfect incompressible fluids.

Random Variable (Aleatory)

Definitions, observations and main properties

RANDOM VARIABLE (ALEATORY)

A FIRST DEFINITION:

$$X : S \longrightarrow A \subseteq \mathbb{R} \begin{cases} A & \text{finite set or infinite countable set} \Rightarrow \boxed{\text{discrete a.v.}} \\ A & \text{continuous set} \Rightarrow \boxed{\text{continuous a.v.}} \end{cases}$$

discrete a.v. : $P(X=x) = f(x)$ PROBABILITY FUNCTION $\begin{cases} f(x_i) \geq 0 \\ \sum_i f(x_i) = 1 \end{cases}$

$\underbrace{P(X \leq x) = F(x)}$ CUMULATIVE DISTRIBUTION FUNCTION $F(x) = \sum_{x_i \leq x} f(x_i)$

F (x) is also called distribution function of the a.v. X (denoted as X~F) and it represents the probability that X adopts a value ≤ x.
This definition is also valid for continuous a.v.
Additionally, note that all probability-related problems involving an a.v. can be solved in terms of its distribution function F.

Definition

An aleatory variable is therefore a number that is assigned to each point of the sample space i.e., to each of the possible outcomes of an aleatory experiment, by means of a specific rule.

CHARACTERISTIC INDEXES OF A RANDOM VARIABLE

EXPECTATION or EXPECTED VALUE

(MEAN)

IT IS ALSO DEFINED AS: EXPECTATION, MEAN OR MATHEMATICAL EXPECTATION

IT IS A POSITION INDEX (it represents the expected value that can be obtained in a large number of tests)

$$E(X) = \begin{cases} \sum_i x_i P(X = x_i) = \sum_i x_i f(x_i) & \text{if DISCRETE A.V.} \\ \int_{-\infty}^{+\infty} x f(x)\, dx & \text{if CONTINUOUS A.V.} \end{cases}$$

it can be $\lesseqgtr 0$

it is often expressed with μ
(in **measure theory** μ is referred to as MEASURED VALUE)
→ where X a.v. is a measurement of a physical parameter

VARIANCE

IT IS A VARIABILITY INDEX (it indicates how "dispersed" the values of the a.v. relatively to its mean value are)

$$Var(X) = \begin{cases} \sum_i (x_i - \mu)^2 P(X = x_i) = \sum_i (x_i - \mu)^2 f(x_i) & \text{if DISCRETE A.V.} \\ \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)\, dx & \text{if CONTINUOUS A.V.} \end{cases}$$

it can be ≥ 0

N.B.:

$$\sigma(X) = \sqrt{Var(X)}$$

STANDARD DEVIATION OF A.V. X ⇒ ROOT-MEAN-SQUARE DEVIATION ⇒ in measure theory it is referred to as **MEASUREMENT UNCERTAINTY**

WE NOTICE THAT:

- $Var(X) = E\left((X-\mu)^2\right)$
- $Var(X) = E(X^2) - \mu^2$ that is: $Var(X) = E(X^2) - E(X)^2$
- $Var(aX) = a^2\, Var(X)$